Journal of Engineering Physics and Thermophysics, Vol. 81, No. 6, 2008

IMPURITY TRANSFER IN THE CASE OF CONVECTION OVER A THERMALLY INHOMOGENEOUS HORIZONTAL SURFACE

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UDC 532.529.2:551.558:551.510.42

Stationary disturbances introduced by convection (thermal circulations) over a thermally inhomogeneous horizontal surface into the background vertical distribution of a passive impurity have been investigated analytically in the linear approximation. It has been found that in the case of a strong stable stratification where vertical motions are suppressed and thermal circulations are "pressed" against the surface impurity-concentration, disturbances produced by them can, nonetheless, penetrate to high levels of the order of horizontal scales of thermal inhomogeneities. Even a very weak convection which has a slight effect on heat transfer and on the background temperature profile, can substantially influence the transfer of a slowly diffused impurity. It is noteworthy that the results in certain situations can be highly sensitive to the details of the boundary conditions at the lower boundary.

Introduction. If a liquid (gaseous) medium is over a nonuniformly heated (cooled) horizontal surface, this means the presence of horizontal pressure gradients (the column weight in this case is dependent on the horizontal coordinates) and hence horizontal flows (thermal circulations). This form of convection existing in stratifications as stable as desired plays an important role in geophysical applications. The horizontal inhomogeneity of the above flows, as is seen from continuity considerations, means the additional occurrence of vertical motions. Therefore, thermal circulations can make an appreciable contribution, in particular, to the transfer of impurities in the ground layer of air. The related theoretical problems are fairly complex and variable; usually, they allow only a numerical solution. In the presence of the background stratification of the impurity (e.g., moisture in the atmosphere or contaminants in the open pit), the problem becomes nontrivial even in the linear approximation, since the term $w\partial s/\partial z$ turns out to be nonzero. In the present work, this approximation enables us to obtain the solution in closed analytical form with quite unexpected results.

Formulation of the Problem. We consider stationary convection (thermal circulations) in a semiinfinite layer (stratified by the temperature and concentration of a passive impurity) of a medium over a thermally inhomogeneous horizontal surface $z = 0$ (the z axis is directed vertically upward). A linearized stationary system of the equations of hydrothermodynamics and transfer of the passive impurity in the Boussinesq approximation has the form [1, 2]

$$0 = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{v} + g\alpha T\mathbf{e}_z\,, \quad \nabla\mathbf{v} = 0\,, \quad \gamma_T\mathbf{v}\cdot\mathbf{e}_z = \kappa\nabla^2 T\,, \quad \gamma_s\mathbf{v}\cdot\mathbf{e}_z = \chi\nabla^2 s\,. \tag{1}$$

Here, ν, κ, and χ are the exchange coefficients for the corresponding substances (as applied to transfer in the atmosphere or basins; the effective coefficients of turbulent exchange may be meant, too). Background stratification is assumed to be stable or neutral ($\gamma_T \geq 0$).

For the sake of simplicity, we restrict our consideration to a two-dimensional problem (in the x, z plane) containing qualitatively new basic results. We assume that, in addition to the nonflow $z = 0$ and adhesion conditions, the disturbances of the vertical heat flux which are harmonically dependent on the horizontal coordinate x

$$u = w = 0\,, \quad c_p\rho_0\kappa\frac{\partial T}{\partial z} = Q\cos(kx) \quad \text{for} \quad z = 0\,. \tag{2}$$

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1062-0125/08/8106-1201

are prescribed on the horizontal surface $z = 0$. For the disturbance of the impurity concentration, we assume that the quite general boundary condition of the 3rd kind

$$\frac{ds}{dz} = \frac{s}{h} \quad \text{for} \quad z = 0 . \tag{3}$$

is observed. All disturbances at a large distance from the underlying surface (for $z \to \infty$) are assumed to be attenuated.

We explain the physical meaning of condition (3) with the example where steam acts as the impurity in the atmosphere (in situations where the buoyancy of the steam can be disregarded). Let the background humidity of air diminish with altitude (background vertical concentration gradient $\gamma_s < 0$). A drier air is transferred to the surface $z = 0$ from above in the region of descending motions. Not only does the vertical humidity gradient remain negative at the surface but it also grows in absolute value: $ds/dz \le 0$. The inflow of the drier air from above leads to a drying of the surface, so that we have $s\big|_{z=0} \le 0$ and the proportionality factor in (3) is $h \ge 0$. If the surface is fairly moist and the mentioned drying is insignificant, we have $h \to +0$. This limiting case corresponds to the boundary condition of the 1st kind. In geophysics, consideration is also given to the opposite cases where the flow of a substance on the surface is considered to be fixed, so that $ds/dz\big|_{z=0} = 0$ and $h \to \infty$ (boundary condition of the 2nd kind). Therefore, it is not unreasonable to analyze the intermediate situations $0 < h < \infty$, too.

Solution. Similarly to a number of works (see, e.g., [1–3] and the references therein), we seek solutions that are also harmonically dependent on the horizontal coordinate, e.g.:

$$w(x, z) = W(z) \cos(kx) . \tag{4}$$

Eliminating all the unknowns, except for w, from (1), we obtain the equation

$$\left(\frac{d}{dz^2} - k^2\right)^3 W = k^6 \mathrm{R} W , \quad \mathrm{R} = \frac{N^2}{\nu\kappa k^4} \ge 0 , \quad N = (\alpha g \gamma_T)^{1/2} . \tag{5}$$

The solution of Eq. (5) is found in a standard manner in the form of the linear combination of exponents $\exp(q_i kz)$. Of the six roots of the characteristic equation

$$(q^2 - 1)^3 = \mathrm{R} \tag{6}$$

it is only three that have a negative real part. The corresponding exponents satisfy the attenuation condition for $z \to \infty$; the coefficients of the remaining ones should be set equal to zero. Therefore, the solution for vertical velocity can be represented in the form

$$W(z) = \sum_{i=1}^{3} C_i \exp(q_i kz) , \quad \mathrm{Re}\, q_i < 0 ; \tag{7}$$

$$q_1 = -(1 + \mathrm{R}^{1/3})^{1/2} , \quad q_{2,3} = \left[1 + \mathrm{R}^{1/3} \exp\left(\pm \frac{2}{3}\pi i\right)\right]^{1/2} . \tag{8}$$

In the general case the explicit form of the solution is quite cumbersome. Below, we restrict our consideration to two cases of the greatest interest: those of neutral and fairly strong stable stratification.

Case of the Absence of Stratification, $\gamma_T = 0$. In the approximation in question, we can represent system (1) as

$$0 = -\frac{\partial \Phi}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right), \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 ; \tag{9}$$

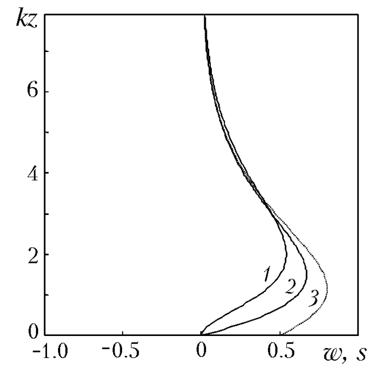

Fig. 1. Dimensionless vertical profiles of disturbances in neutral background temperature stratification over the heated region of the surface (for $x = 0$): 1) vertical velocity normalized to $\alpha g|Q|/(8c_p\rho_0\kappa\nu k^3)$; 2 and 3) impurity-concentration disturbances normalized to $\alpha g\gamma_s Q/(32c_p\rho_0\kappa\nu\chi k^5)$ for values of the parameter hk of 0 and 1 respectively.

$$0 = -\frac{\partial\Phi}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) + g\alpha T, \quad 0 = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right); \tag{10}$$

$$\gamma_s w = \chi\left(\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial z^2}\right), \tag{11}$$

where $\Phi = p/\rho_0$. The solution of the close convection problem (with other boundary conditions) has been given in [3]. Here it has the form

$$T = -\frac{Q}{c_p\rho_0\kappa k}\exp(-kz)\cos(kx), \tag{12}$$

$$u = \frac{\alpha g Q}{8c_p\rho_0\kappa\nu k^2} z(2 - kz)\exp(-kz)\sin(kx), \tag{13}$$

$$w = -\frac{\alpha g Q}{8c_p\rho_0\kappa\nu k} z^2 \exp(-kz)\cos(kx), \tag{14}$$

$$p = \frac{\alpha g Q}{4c_p\kappa k^2}(3 - 2kz)\exp(-kz)\cos(kx),$$

$$s = \frac{\alpha g\gamma_s Q}{32c_p\rho_0\kappa\nu\chi k^5}\left(\frac{hk}{1 + hk} + kz + \frac{2}{3}k^2z^2\right)\exp(-kz)\cos(kx). \tag{15}$$

Figure 1 exemplifies the dimensionless vertical profiles of disturbances for $Q < 0$ on the vertical $x = 0$, i.e., over the region of the maximum heat flux from the horizontal surface $z = 0$. The disturbances penetrate into the medium to altitudes of the order of the horizontal scale of thermal inhomogeneities $L = 2\pi/k$. Circulation cells (shafts) with ascending motions over more heated regions with an aspect ratio (horizontal-to-vertical-scale ratio) of the order of unity occur over the thermally inhomogeneous surface. The amplitudes of disturbances of the velocity components u

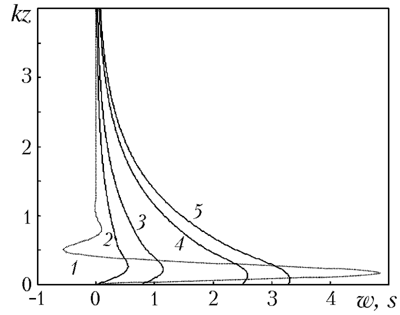

Fig. 2. Profiles of disturbances in strong stable background temperature stratification ($R = 3{\cdot}10^6$) over the heated region of the surface: 1) vertical velocity normalized to $k|Q|R^{1/6}/(c_p\rho_0\gamma_T)$; 2–5) impurity-concentration disturbances normalized to $\gamma_s Q/(c_p\rho_0\chi\gamma_T k R^{1/6})$ for values of the parameter hk of 0, 0.3, 3 and 100 respectively.

and w are of the order of $\alpha g Q/(8c_p\rho_0\kappa\nu k^3)$, and those of disturbances of the impurity concentrations are of the order of $\alpha g\gamma_s Q/(32 c_p\rho_0\kappa\nu\chi k^5)$ (noteworthy is the strong dependence on the horizontal scales of thermal inhomogeneities, i.e., on the k value). When $Q < 0$ and $\gamma_s < 0$ (the background concentration diminishes with altitude) the concentration disturbances in the region of ascending motions are positive, as could be expected. The concentration disturbance increases with the dimensionless parameter hk (reciprocal of the Biot number [4]). This is easily explained, since a decrease in h corresponds to a more rigid fixation of the value of concentration on the surface $z = 0$.

Case of Strong Stable Stratification, $\gamma_T > 0$ and $R^{1/6} >> 1$. This corresponds to situations where the stable density stratification largely determines the structure of disturbances occurring in the medium. In such cases we can disregard unity compared to other terms in expressions (6) and (8):

$$q_1 \approx -R^{1/6}, \quad q_{2,3} \approx -R^{1/6}\exp\left(\pm\frac{1}{3}\pi i\right)$$

and the general solution is noticeably simplified. A large factor $R^{1/6}$ is present in the exponents in (7) (and in the expressions for u and pressure and temperature disturbances), so that the disturbances of the mentioned quantities are rapidly attenuated with altitude compared to their characteristic horizontal scale L. In other words, the aspect ratio of these disturbances is much greater than unity: unlike the previous case, the emerging convective cells are strongly "pressed" against the horizontal surface $z = 0$, since stable stratification prevents vertical motions. In particular, the approximate solution for vertical velocity has the form

$$w \approx \frac{kQR^{1/6}}{c_p\rho_0\gamma_T}\left[\frac{2}{\sqrt{3}}\cos\left(\frac{\sqrt{3}}{2}R^{1/6}kz + \frac{\pi}{6}\right) - \exp\left(-\frac{1}{2}R^{1/6}kz\right)\right]\exp\left(-\frac{1}{2}R^{1/6}kz\right)\cos(kx)\,. \tag{16}$$

The solution of the equation for the concentration of the impurity (11) in the same approximation has the form

$$s \approx \frac{Q\gamma_s}{c_p\rho_0\chi\gamma_T kR^{1/6}}\cos(kx)$$

$$\times\left[\frac{2 + hkR^{1/6}}{1 + hk}\exp(-kz) - \exp(-R^{1/6}kz) - \frac{2}{\sqrt{3}}\exp\left(-\frac{1}{2}R^{1/6}kz\right)\sin\left(\frac{\sqrt{3}}{2}R^{1/6}kz + \frac{\pi}{3}\right)\right]. \tag{17}$$

Analysis of the Solution and Discussion of Results. Compared to (14) and (15), the amplitude of the vertical velocity diminishes in the presence of stratification as $R^{-5/6}$, whereas the amplitude of the impurity concentration decreases as $R^{-7/6}$ (for low values of the parameter hk). The dimensionless vertical profiles of the disturbances are exemplified in Fig. 2. As could be expected, thermal circulations due to the strong stratification are extended horizontally and are confined, according to (16), in a thin (compared to the horizontal scale L) layer with a thickness of the order of $L/\mathrm{R}^{1/6} \sim (k\mathrm{R}^{1/6})^{-1} \sim (\nu\kappa L^2/N^2)^{1/6}$. At first glance, it seems evident that concentration disturbances caused by the transfer of the impurity by these circulations must be confined in this layer, too. The result obtained is all the more nontrivial: a slowly decreasing exponent exp $(-kz)$ appears in (17), unlike (16), in addition to exp $(-\mathrm{R}^{1/6}kz)$-type exponents rapidly decreasing with altitude. This means that the impurity-concentration disturbances go far beyond the mentioned thin layer: they reach levels of $z \sim L$.

Another nontrivial result is the sensitivity of the solution to the lower boundary condition for s with small variations from the often-used condition of the 1st kind (with low h values). The contribution of the slowly decreasing exponent is substantially dependent on condition (3): on the value of the dimensionless parameter hk. If this parameter is less or of the order of $\mathrm{R}^{-1/6} << 1$, the coefficient of the slowly decreasing exponent is of the order of unity. Consequently, this exponent, even if dominant outside the mentioned thin layer containing thermal circulations, has the same order of magnitude as the remaining terms in this layer itself. If we have $hk >> \mathrm{R}^{-1/6}$, this exponent is the basic term in the expression for the concentration disturbance, which thereby not only deeply penetrates into a medium stratified as stable as desired but can also be much more intense that the above-mentioned asymptotics $\mathrm{R}^{-7/6}$. Thus, in the case of a strong stable stratification the propagation of the impurity can substantially be dependent on boundary condition (3) (for the sake of comparison, the dependence on the parameter hk in the solution (15) is much weaker).

These results are easily understood from the following considerations. Stable stratification suppresses temperature and velocity disturbances, which thereby cannot penetrate deep into the medium. But the stratification exerts no direct influence on the diffusion of the appearing disturbances of the impurity concentration (the temperature stratification is not involved in the impurity-diffusion equation), so that these disturbances propagate vertically to levels of $z \sim L$, just as in the case of the absence of stratification. The source of concentration disturbances (circulation cells) in the case of a strong stratification is concentrated in the thin layer; therefore, the boundary conditions at the horizontal boundary $z = 0$ substantially increase in importance.

When the values of the dimensionless parameter hk are low (when the boundary condition is close to the condition of the 1st kind), we easily obtain, from (17), the estimate

$$\left| \left. \frac{1}{\gamma_s} \frac{ds}{dz} \right|_{z=0} \right| \sim \left| \frac{Q}{c_p \rho_0 \chi \gamma_T} \right| .$$

This dimensionless quantity represents the relative disturbance of the diffusion flux of the impurity (at the surface $z = 0$) due to the convection (thermal flows) in question. Its ratio to an analogous quantity for the disturbance of the vertical heat flux is of the order of κ/χ, as is easily checked. This means that even a very weak convection which influences the heat transfer and the background temperature profile can only slightly exert a substantial influence on the transfer of a slowly diffused impurity (situation characteristic of, e.g., salt in seawater, where $\chi << \kappa$.)

In addition to the impurity, circulation cells can transfer such a substance as momentum. If there is background vertical-shear flow directed along the axis of these cells (in the direction of the horizontal y axis), the action of convective circulations on this shear flow is of interest. Such a problem has been considered in [3], where investigation, however, was restricted to the case of neutral stratification. The linearized equation of transfer of the momentum component in the y direction [3] is coincident with the impurity-diffusion equation considered above accurate to the notation. Thus, now we can use the results obtained above for analysis of the disturbances of the background shear flow under the influence of thermal circulations. The qualitatively new result in this case is that the "shallow" circulation cells in question can lead to "deep" (to levels of $z \sim L$) disturbances of the shear flow, just as in the case of impurity transfer.

Conclusions. The linearized problem on disturbances of the background impurity distribution under the influence of convection (thermal circulations) has analytically been solved. It has been found that circulations confined in a thin surface layer, which are characteristic of stable stratification, can give rise to disturbances in the impurity-concentration field even in the linear approximation; the disturbances formed penetrate deep into the stably stratified medium. This is also true of the disturbances introduced into the velocity field of the background shear flow. From the results obtained, it is clear that even a very weak convection which influences the heat transfer and the background temperature profile only slightly is capable of exerting a substantial influence on the transfer of a slowly diffused impurity. Also, it has been established that the results can be highly sensitive to variations from the boundary conditions of the 1st kind. The possibility of the "small" thermal circulations influencing the distribution of substances at relatively high altitudes seems rather important, since this also means the presence of the additional mechanism of horizontal transfer: the impurity risen high in the atmosphere arrives at the field of a stronger background wind.

The author expresses his thanks to N. P. Romanov for stimulating discussions.

This work was carried out with support from the Russian Foundation for Basic Research (project 04-05-64027).

NOTATION

C_i, integration constants, m/sec; c_p, specific heat of the medium, J/(kg·K); $\mathbf{e}_z$, vertical unit vector; g, free-fall acceleration, m/sec^2; h, specified length scale, m; i, imaginary unit; k, wave number, m^{-1}; L, horizontal scale of thermal inhomogeneities, m; N, Brent–Väisälä frequency, sec^{-1}; p, pressure disturbance, Pa; Q, amplitude of the disturbance of the vertical heat flux, W/m^2; q_i, dimensionless roots of the characteristic equation; R, analog of the dimensionless Rayleigh number; s, disturbance of the passive-impurity concentration, m^{-3}; T, temperature disturbance, K; u, components of the velocity in the direction of the horizontal axis, m/sec; $\mathbf{v}$, velocity vector, m/sec; w, component of the velocity in the direction of the vertical axis z, m/sec; W, amplitude function of the vertical velocity, m/sec; x, y, z, horizontal and vertical coordinates, m; α, thermal expansion coefficient of the medium, K^{-1}; γ_T, background vertical temperature gradient, K/m; γ_s, background vertical impurity-concentration gradient, m^{-4}; κ, thermal diffusivity of the medium, m^2/sec; ν, kinematic coefficient of viscosity of the medium, m^2/sec; ρ_0, average (reference) density of the medium, kg/m^3; $\Phi = p/\rho_0$, normalized pressure disturbance, m^2/sec^2; χ, coefficient of transfer of the impurity, m^2/sec. Subscripts: i, number of the root of the characteristic equation; 0, background value.

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